A Democratic Measure of Household Income Growth: Theory and Application to the United Kingdom

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Abstract
This paper develops a price and quantity system of indicators structured round Atkinson's concept of inequality aversion. A democratic indicator of income growth, weighting each household's growth experience equally, is shown to result when Prais' democratic price index is used to deflate the geometric mean of equivalised household income. A welfare interpretation of the democratic indicator of income growth is provided and it is shown that, with heterogeneous but homothetic preferences, the deflator can serve as a common scaling social cost of living index when applied to income as well as to consumption. Application to United Kingdom household data suggests that, over the interval 2005/6-2015/6 democratic real equivalised household income grew by 0.20 per cent per annum while the plutocratic equivalent grew by 0.52 per cent per annum.

Key words: Real Income, Inequality Aversion, Welfare Indicator, Cost of Living

JEL classification: I31, D12, E21

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1 Introduction

In this paper we develop aggregate price and quantity indices which reflect inequality aversion of the type described by Atkinson (1970). We show in particular that, when a nominal aggregate is constructed as the geometric mean of household income and that aggregate is deflated using an appropriate democratic price index which gives equal weight to the expenditure patterns of each household (Prais 1958), the resulting quantity variable is an index whose growth is the arithmetic average of each household's growth in real income. Following Prais (1958) it can therefore be described as a democratic measure of household real income growth with the property that it treats each household's growth experience equally. Newbery (1995) used a similar approach to explore the welfare effects of price changes in Hungary and the United Kingdom but otherwise, as far as we know, the relationship between Atkinson's approach and price and quantity indicators has not been explored.

The need for indicators which, unlike GDP are sensitive to the distribution of resources is well documented. The Stiglitz Commission (Stiglitz, Sen, and Fitoussi 2009) observed that "if inequality increases enough relative to the increase in the average of per capita GDP, then most people can be worse off even though average income is increasing". Jorgenson and Slesnick (2014) following Jorgenson (1990) suggest addressing this issue by basing a welfare measure on consumption rather than income or output; their proposal is to use an econometrically estimated demand system to provide the structure needed to produce a cardinal measure of utility, and they present a welfare indicator for the United States on that basis. In this econometric framework utility is calculated per household, with each household's circumstances reflected in an econometrically-estimated equivalence scale which takes account not only of household composition but also of other factors such as age and location. In a similar vein Oulton (2008) explored a Konüs price index for the United Kingdom, estimating the cost of obtaining the consumption which would deliver a fixed level of utility for a representative household.

A related literature has evolved from the work by Pollak (1981) who explored the development of a social cost of living index. His approach was to calculate the increase in expenditure needed to deliver a particular level of welfare, on the assumption that resources were allocated to optimise the relevant social welfare function. Crossley and

Pendakur (2010) have extended the concept to a constant-scaling social cost of living index (CS-SCOLI); they show how, given knowledge of the actual distribution of consumption, each household's welfare function and the means by which these are aggregated into the relevant social welfare function, it is possible to define the change in the cost of living associated with a given change in prices as the proportionate increase in each household's expenditure required to restore the social welfare function to its previous value. Thus, in contrast to Pollak (1981), they work with reference to the distribution of resources as it actually is. Of course constant scaling may leave some households better off and others worse off if their consumption patterns differ, but for policy purposes such as the adjustment of state benefits it is probably the most relevant approach. Nevertheless, their work, like Jorgenson and Slesnick (2014), requires econometric estimation of a demand system. They make no attempt to develop a cardinal measure of social welfare or any other type of quantity indicator.

Jorgenson and Schreyer (2017) recognise, however, that approaches of this type are unlikely to appeal strongly to statistical offices. They therefore suggest a simplified welfare indicator compiled by adjusting household consumption for household size using a standard equivalence scale such as the square root scale (Organisation for Economic Co-operation and Development 2011). After scaling and deflation Jorgenson and Schreyer (2017) suggest combining the household figures using Atkinson's inequality aversion approach[1] with the inequality aversion parameter set to a value designed, for the United States at least, to approximate to the median household. They note that there is an issue with deflation because different households buy different things, and suggest splitting households into quintiles defined over equivalent consumption, with deflators reflecting the spending patterns of those quintiles. They do not, however, explore the price/quantity/value inter-relationship arising from their measures.

This measure is structured round consumption rather than income. There are two reasons for this. Slesnick (1998) observes that households with temporarily low incomes will maintain their consumption, while those with temporarily high incomes will be high savers; consumption may therefore be a better indicator of life-time income and thus life-time resources than is current income. Secondly, a focus on consumption makes it straightforward to draw on the wide body of consumer theory providing an underpin-

[1] They also present an alternative adjustment for inequality developed by Jorgenson and Slesnick (1983) but which is not used as widely as Atkinson's approach.

ning for the approaches suggested by Crossley and Pendakur (2010) and Jorgenson and Slesnick (2014). On the other hand there have been a number of attempts to give a welfare interpretation to income rather than consumption. At an aggregate level Weitzman (1976) suggested that net national income, measured in consumption terms, was, subject to a linear approximation, equal to sustainable consumption and thus could be seen as a welfare indicator. Sefton and Weale (2006) developed Weitzman's analysis, first on an individual and then on an aggregate basis. They showed that, in an economy in which capital was being accumulated or decumulated, it was not possible to equate income and sustainable consumption but that net saving multiplied by the marginal utility of money was nevertheless equal to the increase in lifetime utility in which it gave rise.

Here we build on that result to provide an interpretation of growth in real income for each household in terms of an associated increase in life-time utility. We show that, with a democratic measure of real income growth, each household's increase in life-time utility is measured with respect to its own marginal utility of consumption while the conventional measure is defined with respect to average marginal utility of consumption. These results require that households have homothetic but not necessarily homogenous preferences. While it is clear that household preferences cannot be represented by a single homothetic preference function, Redding and Weinstein (2016) argue that movements in household consumption patterns are better explained by preference shifts than by more traditional aggregate preference functions. Further, their analysis of scanner data from the United States of America suggests that these preference shifts are entirely compatible with the requirements of homotheticity.

Finally, again given the assumption of homothetic but heterogeneous preferences possibly affected by preference shocks, we show that the democratic price index and quantity growth indicator emerge if each household's utility is logarithmic in its deflated consumption aggregate. We demonstrate that, in these circumstances, the democratic Divisia price index is a CS-SCOLI. We show further that, to a first order approximation, welfare accruing can coherently be represented as a utility function in terms of income; the democratic Divisia price index again emerges as a CS-SCOLI and the associated democratic measure of income growth represents the rate of growth of an index of social welfare. Thus, the same system of price and quantity indices is derived whether one works heuristically from Atkinson inequality aversion with an aversion parameter of one or from a logarithmic household utility function. Our theoretical analysis is set out entirely in terms of continuous time. Application requires chain-linking of the type which is now widespread.

This paper proceeds as follows. Section 2 describes our notation. Section 3 explains the connection between democratic price and quantity indices and Atkinson's concept of inequality aversion. Section 4 provides a welfare interpretation of democratic income and section 5 explains the relationship with constant scaling cost of living indices. Section 6 presents results for the UK and section 7 concludes.

2 Notation

We use lower case letters to refer to individual households and upper case letters to refer to aggregates. $\boldsymbol{\pi}_t$ is the vector of prices in the economy at time t, with the jth element, π_{jt} indicating the price of good j and $\mathbf{c}_{it}$ is the vector of equivalised consumption of household i with c_{ijt} the consumption of good j. y_{it} is its equivalised money income and x_{it} is its equivalised money spending on consumption goods. $u_i(\boldsymbol{c}_{it})$ is the utility derived from equivalised consumption by household i, and $z_i(\boldsymbol{\pi}_t, x_{it})$ is indirect utility from equivalised consumption x_{it}; $\boldsymbol{\omega}_{it}$ is the vector of expenditure shares measured as a fraction of total consumption, and ω_{ijt} is the share of expenditure by household i on good j. p_{it} is a consumption price index for household i and $q_{it} = y_{it}/p_{it}$ is its equivalised real income. r^*_t is the money rate of interest and r_{it} is the real interest rate faced by household i; different households have different real interest rates because they have different consumption patterns. ρ is the degree of inequality aversion, and σ is the intertemporal elasticity of substitution. $Y_t(\rho), P_t(\rho)$ and $Q_t(\rho)$ are aggregate variables representing income, prices and quantities defined precisely as they are introduced. Our analysis is in continuous time; we use the symbol Δ to indicate the difference between one continuous time path and another.

In this paper we do not discuss equivalisation but simply assume that all income and consumption variables have been adjusted for household composition using a standard equivalence scale. References to household income and consumption are thus shorthand for references to equivalised income and consumption.

3 Atkinson Inequality Indices and a Democratic Measure of Income Growth

3.1 Derivation of Quantity Indices

Atkinson (1970) developed the concept of inequality aversion. For a population of N households, with household i having an income of y_{it} in period t. Atkinson's inequality-adjusted measure of average income is defined as

$$\begin{aligned} Y_t(\rho) &= \frac{1}{N}(\sum_{i=1}^{N} y_{it}^{1-\rho})^{\frac{1}{1-\rho}} \text{ with } \rho \geq 0 \text{ and } \rho \neq 1 \\ Y_t(1) &= \sqrt[N]{\prod_{i=1}^{N} y_{it}} \quad \rho = 1 \end{aligned} \tag{1}$$

Each household's income is the product of its price index p_{it} and its real income q_{it}; $y_{it} = p_{it}q_{it}$.

Suppose we now express $Y_t(\rho)$ as the product of aggregate price and quantity measures

$$Y_t(\rho) = P_t(\rho)Q_t(\rho) = \frac{1}{N}(\sum_{i=1}^{N} p_{it}^{1-\rho} q_{it}^{1-\rho})^{\frac{1}{1-\rho}} \text{ with } \rho \geq 0 \text{ and } \rho \neq 1 \tag{2}$$

We now take logarithms and differentiate with respect to time, with the expression valid even if $\rho = 1$

$$\begin{aligned} \frac{\dot{P}_t(\rho)}{P_t(\rho)} + \frac{\dot{Q}_t(\rho)}{Q_t(\rho)} &= \frac{\sum_{i=1}^{N} \left(\frac{\dot{p}_{it}}{p_{it}} p_{it}^{1-\rho} q_{it}^{1-\rho} + \frac{\dot{q}_{it}}{q_{it}} p_{it}^{1-\rho} q_{it}^{1-\rho}\right)}{\sum_{i=1}^{N} p_{it}^{1-\rho} q_{it}^{1-\rho}} \\ &= \frac{\sum_{i=1}^{N} \left(\frac{\dot{p}_{it}}{p_{it}} + \frac{\dot{q}_{it}}{q_{it}}\right) y_{it}^{1-\rho}/Y_t^{1-\rho}(\rho)}{N} \end{aligned} \tag{3}$$

This makes it natural to define the growth in the price and quantity indices as

$$\frac{\dot{P}_t(\rho)}{P_t(\rho)} = \frac{\sum_{i=1}^{N} \frac{\dot{p}_{it}}{p_{it}} y_{it}^{1-\rho}/Y_t^{1-\rho}(\rho)}{N} \tag{4}$$

and

$$\frac{\dot{Q}_t(\rho)}{Q_t(\rho)} = \frac{\sum_{i=1}^{N} \frac{\dot{q}_{it}}{q_{it}} y_{it}^{1-\rho}/Y_t^{1-\rho}(\rho)}{N} \tag{5}$$

We can now see very clearly that, in the special case where there is no inequality aversion ($\rho = 0$) the growth rates of the aggregate price and quantity indices are the growth rates experienced by the individual households weighted together by their shares in total money income. On the other hand, if $\rho = 1$, then the price and quantity indices are simply the arithmetic averages of the growth experiences of individual households. Each household has an equal influence on the growth of the aggregates, and in that sense $\dot{Q}_t(1)/Q_t(1)$ is, following Prais (1958) who described the price index with $\rho = 1$ as democratic, a democratic measure of income growth. In contrast $\dot{Q}_t(0)/Q_t(0)$ because it weights the household experiences by their income levels, can be described as a plutocratic measure.

There remains the question of how to calculate either the price or the quantity index for each household. We assume that it is easier to measure prices than volumes, and that changes in quantity indices have to be derived from changes in values and changes in price indices, both individually and at an aggregate level. To proceed it is necessary to consider the "real" price of good j to household i, π_{jt}/p_{it}. The price index, p_{it} is defined by the condition that, when the household has optimised its consumption, in the light of the prevailing prices, the cost of the consumption basket in real prices is constant as nominal prices change:

$$\sum_j c_{ijt} \frac{d\left(\pi_{jt}/p_{it}\right)}{dt} = \sum_j c_{ijt} \frac{p_{it}\dot{\pi}_{jt} - \pi_{jt}\dot{p}_{it}}{p_{it}^2} = 0 \tag{6}$$

or

$$\sum_j c_{ijt} \frac{\dot{\pi}_{jt}}{p_{it}} = \frac{\dot{p}_{it}}{p_{it}^2} \sum_j c_{ijt}\pi_{jt} \tag{7}$$

giving

$$\frac{\dot{p}_{it}}{p_{it}} = \frac{\sum_j c_{ijt}\dot{\pi}_{jt}}{\sum_j c_{ijt}\pi_{jt}} = \sum_j \omega_{ijt} \frac{\dot{\pi}_{jt}}{\pi_{jt}} \tag{8}$$

This, of course, is the equation for the growth of the Divisia price index specific to household i.

Combining this with equation 4, we can write

$$\begin{aligned} \frac{\dot{P}_t(\rho)}{P_t(\rho)} &= \frac{\sum_{i=1}^N \sum_j \omega_{ijt} \frac{\dot{\pi}_{jt}}{\pi_{jt}} y_{it}^{1-\rho}/Y_t^{1-\rho}(\rho)}{N} \\ &= \frac{\sum_j \frac{\dot{\pi}_{jt}}{\pi_{jt}} \sum_{i=1}^N \omega_{ijt} y_{it}^{1-\rho}/Y_t^{1-\rho}(\rho)}{N} \end{aligned} \tag{9}$$

The growth of the aggregate price index is itself the weighted sum of the growth rates of the individual prices. The weights depend on the consumption patterns of the individual households themselves weighted together by each household's inequality-adjusted income relative to inequality adjusted average income. In the special case where $\rho = 1$ we have

$$\frac{\dot{P}_t(1)}{P_t(1)} = \frac{\sum_j \frac{\dot{\pi}_{jt}}{\pi_{jt}} \sum_{i=1}^N \omega_{ijt}}{N}. \tag{10}$$

In this case, then, the weights applied to the growth of each price are the arithmetic averages of each household's expenditure shares. This is exactly the Divisia form of the democratic price index suggested by Prais (1958). In other cases, however, it should be noted that the weights depend on income rather than on consumption.

Growth in the aggregate quantity index is then defined as

$$\frac{\dot{Q}_t(\rho)}{Q_t(\rho)} = \frac{\dot{Y}_t(\rho)}{Y_t(\rho)} - \frac{\sum_j \frac{\dot{\pi}_j}{\pi_j} \sum_{i=1}^N \omega_{ijt} y_{it}^{1-\rho} / Y_t^{1-\rho}(\rho)}{N} \tag{11}$$

Integrating up and using subscripts to indicate time, we can write

$$\log \frac{Q_t(\rho)}{Q_{t_0}(\rho)} = \log \frac{Y_t(\rho)}{Y_{t_0}(\rho)} - \int_{t_0}^t \frac{\sum_j \frac{\dot{\pi}_{j\tau}}{\pi_{j\tau}} \sum_{i=1}^N \omega_{ij\tau} y_{i\tau}^{1-\rho} / Y_\tau^{1-\rho}(\rho)}{N} d\tau \tag{12}$$

In practice of course it is not possible to construct the Divisia price index required. But we could calculate $P_t(\rho)$ as a chain-linked price index, making it possible to define

$$\frac{Q_t(\rho)}{Q_{t_0}(\rho)} = \frac{Y_t(\rho)}{Y_{t_0}(\rho)} \frac{P_{t_0}(\rho)}{P_t(\rho)} \tag{13}$$

The assumption that values can be uniquely decomposed into quantity and price indices is valid only when behaviour is homothetic (see for example Samuelson and Swamy (1974) and Balk (1995)). Unless income elasticities of demand are equal to one, the resulting chain-linked price and quantity indices are path-dependent.

The issue could be avoided by, for example replacing the expenditure shares in (10) by expenditure shares in a base period, ω_{ijt_0} giving a Laspeyres democratic price index. But this does not really resolve the problem; weights become stale as time passes and chain-linking has always been used as a means of resolving this. The 1993 *System of National Accounts* specified that chain-linking should take place annually rather than intermittently, and most price and volume decompositions now follow that practice.

It should be stressed, nevertheless that, for our measure to avoid path dependence we do not require homogeneity of expenditure patterns across households. Furthermore, as we show in appendix A, for a constant elasticity of substitution demand system, we do not require that the demand parameters should be invariant over time although parameter shocks cannot be unrestricted. As noted in the introduction, Redding and Weinstein (2016), on the basis of analysis of scanner data for fifty thousand households in the United States, argue that patterns of demand are better explained by homothetic preference functions subject to demand shocks than by more conventional demand systems; they suggest that the preference shocks meet the requirement set out in appendix A for the Divisia price index to be path-independent.

3.2 Implications for Measuring Changes to Inequality over Time

The Atkinson Inequality index is given as

$$A_t(\rho) = 1 - Y_t(\rho)/Y_t(0) \tag{14}$$

Conventionally changes to this over time can be used to indicate changes to inequality, with the result of course depending on the degree of inequality aversion assumed.

Movements in nominal variables are, however, not good indicators of movements in living standards when households with different incomes have different consumption patterns. If the prices of goods bought disproportionately by the poor increase in price faster than those bought disproportionately by the rich, then even if the nominal distribution is unchanged, inequality can be said to have increased. This suggests that, to compare changes in equality between periods 0 and t it is better to look at

$$A_t^q(\rho) - A_{t_0}^q(\rho) = Q_t(\rho)/Q_t(0) - Q_{t_0}(\varrho)/Q_{t_0}(0) \tag{15}$$

with $Q_{t_0} = Y_{t_0}$ if t_0 is the base period for the price and quantity indices.

4 A Welfare Interpretation

It is possible to give a welfare interpretation to the growth of the inequality-adjusted quantity index when saving is the only source of income growth. To do this, however, it is necessary to make the assumption that each household has homothetic preferences. This is, of course, a much weaker assumption than the proposition that there is a single representative consumer with homothetic preferences. Further, as mentioned above, shifts in preference parameters are, subject to the constraint presented in Appendix A, not ruled out.

4.1 Income, Saving and Welfare of an Individual Household

Sefton and Weale (2006) explored the concept of income in a general equilibrium and the relationship between that and welfare in an economy where households have homothetic preferences. They considered an inter-temporally optimising household, i, which consumes a vector of consumption goods, $\mathbf{c}_{it}$ with utility function $u_i(\mathbf{c}_{it})$ in period t and a discount rate of θ, giving an inter-temporal welfare function of $\int_t^\infty u_i(\boldsymbol{c}_{i\tau})e^{-\theta\tau}d\tau$.

They showed (p. 226) that, when resources are allocated efficiently inter-temporally

$$\frac{d}{dt}\int_t^\infty u_i(\boldsymbol{c}_{i\tau})e^{-\theta\tau}d\tau = p_{it}\frac{\partial z_{it}}{\partial x_{it}}(\int_t^\infty (r_{i\tau}\boldsymbol{\pi}'_\tau\boldsymbol{c}_{i\tau}e^{-\int_t^\tau r_{i\nu}d\nu}/p_{i\tau})d\tau - \boldsymbol{\pi}'_t\boldsymbol{c}_{it}/p_{it}) \tag{16}$$

Here $r_{i\tau}$ is the real rate of return faced by household i defined as the nominal market rate of return less the rate of growth of the household-specific Divisia price index, $p_{i\tau}$.

Further, if saving is the only source of real income growth, the real income, q_{it} of the household satisfies the differential equation

$$\dot{q}_{it} = r_{it}(q_{it} - \boldsymbol{\pi}'_t\boldsymbol{c}_{it}/p_{it}) \tag{17}$$

The growth in real income is equal to the flow of saving multiplied by the real rate of return. This equation can be extended to allow for a second and quite general source of household-specific real income growth h_{it} to give a general equation for household income growth of

$$\dot{q}_{it} = r_{it}(q_{it} - \boldsymbol{\pi}'_t\boldsymbol{c}_{it}/p_{it}) + h_{it} \tag{18}$$

The analysis proceeds on the assumption that there is perfect foresight; the current and future exogenous contributions to income shocks are known.

Using $e^{-\int_t^\tau r_{i\nu} d\nu}$ as an integrating factor we write

$$d\frac{q_{i\tau}e^{-\int_t^\tau r_{i\nu}d\nu}}{d\tau} = (h_{i\tau} - r_{i\tau}\boldsymbol{\pi}_t'\boldsymbol{c}_{i\tau}/p_{i\tau})e^{-\int_t^\tau r_{i\nu}d\nu} \tag{19}$$

Now integrating from t to infinity

$$\left[q_{i\tau}e^{-\int_t^\tau r_{i\nu}d\nu}\right]_t^\infty = \int_t^\infty (h_{i\tau} - r_{i\tau}\boldsymbol{\pi}_\tau'\boldsymbol{c}_{i\tau}/p_{i\tau})e^{-\int_t^\tau r_{i\nu}d\nu}d\tau \tag{20}$$

giving

$$q_{it} + \int_t^\infty h_{i\tau}e^{-\int_t^\tau r_{i\nu}d\nu}d\tau = \int_t^\infty \frac{(r_{i\tau}\boldsymbol{\pi}_\tau'\boldsymbol{c}_{i\tau})\, e^{-\int_t^\tau r_{i\nu}d\nu}}{p_{i\tau}}d\tau \tag{21}$$

We can now combine this with equation (16) to give

$$\frac{\frac{d}{dt}\int_t^\infty u_i(\boldsymbol{c}_{i\tau})e^{-\theta\tau}d\tau}{p_{it}\frac{\partial z_{it}}{\partial x_{it}}} - \int_t^\infty h_{i\tau}e^{-\int_t^\tau r_{i\nu}d\nu}d\tau = (q_{it} - \boldsymbol{\pi}_t'\boldsymbol{c}_{it}/p_{it}) \tag{22}$$

and thus

$$\begin{aligned}\frac{\dot{q}_{it}}{q_{it}} &= \frac{r_{it}}{p_{it}q_{it}\partial z_{it}/\partial x_{it}}\frac{d}{dt}\int_t^\infty u_i(\boldsymbol{c}_{i\tau})e^{-\theta\tau}d\tau + \frac{h_{it} - r_{it}\int_t^\infty h_{i\tau}e^{-\int_t^\tau r_{i\nu}d\nu}d\tau}{q_{it}} \\ &= \frac{r_{it}}{y_{it}\partial z_{it}/\partial x_{it}}\frac{d}{dt}\int_t^\infty u_i(\boldsymbol{c}_{i\tau})e^{-\theta\tau}d\tau + \frac{h_{it} - r_{it}\int_t^\infty h_{i\tau}e^{-\int_t^\tau r_{i\nu}d\nu}d\tau}{q_{it}}\end{aligned} \tag{23}$$

The first term in this equation includes the increase in the discounted sum of future utility measured relative to income valued at its marginal utility; this can be thought of as the proportionate increase in a capital sum. Mutliplying by the real interest rate yields the dividend on this capital sum. The second component is positive if the current residual component of income growth, h_{it}, is larger than the dividend on all future discounted increases, $r_{it}\int_t^\infty h_{i\tau}e^{-\int_t^\tau r_{i\nu}d\nu}$, and negative if it falls short. In the special case where the two are equal the rate of growth of real income is proportional to the dividend from the growth in future utility. We set $s_{it} = \frac{h_{it}-r_{it}\int_t^\infty h_{i\tau}e^{-\int_t^\tau r_{i\nu}d\nu}d\tau}{q_{it}}$ in order to simplify the subsequent exposition. This represents the instantaneous exogenous contribution to growth relative to the dividend earned on all discounted future increases.

4.2 Aggregation using Income Inequality Aversion

From this identity for an individual household we are now able to show how the growth rates in measures of real income produced with varying degrees of inequality aversion weight together the growth in the utility of individual households and the exogenous term.

$$\frac{\dot{Q}_t(\rho)}{Q_t(\rho)} = \left(\sum_{i=1}^{N} \left(y_{it}^{1-\rho} / Y_t^{1-\rho}(\rho) \right) \left\{ \frac{r_{it}}{y_{it} \frac{\partial z_i}{\partial x_{it}}} \frac{d}{dt} \int_t^{\infty} u_i(\boldsymbol{c}_{i\tau}) e^{-\theta\tau} d\tau + s_{it} \right\} \right) / N \tag{24}$$

If $\rho = 1$, then we have

$$\frac{\dot{Q}_t(1)}{Q_t(1)} = \left(\sum_{i=1}^{N} \left\{ \frac{r_{it}}{y_{it} \frac{\partial z_i}{\partial x_{it}}} \frac{d}{dt} \int_t^{\infty} u_i(\boldsymbol{c}_{i\tau}) e^{-\theta\tau} d\tau + s_{it} \right\} \right) / N \tag{25}$$

and we have, not surprisingly the arithmetic average of the dividend on the growth in utility measured with reference to the marginal utility of each household's actual income plus the arithmetic average of the exogenous contributions measured relative to the dividend on their discounted future value. This average can be close to zero even when the individual components are not.

We noted earlier that, except when $\rho = 1$, the Atkinson measure of inequality aversion applied to money income and its decomposition into prices and quantities results in a price index in which individual household price indices are weighted with reference to household shares in total income instead of shares in total consumption. Rather, therefore, than explore the interpretation of an income-weighted plutocratic measure of individual growth, we move straight to an interpretation derived when household price indices are weighted by consumption shares, since this, apart from the approximations arising from chain linking, corresponds to what emerges when mean income is deflated by a conventional consumption deflator.

4.3 The Plutocratic Measure evaluated when Household Indices are given Consumption rather than Income Weights

Kehoe and Levine (1990) proved that the equilibrium path of a competitive economy with infinitely-lived households could be shown to result from the optimisation of a social welfare function constructed as the weighted sum of each household's inter-temporal welfare[2].

$$U = \sum_i \alpha_i \int_t^\infty u_i(\boldsymbol{c}_{i\tau}) e^{-\theta\tau} d\tau \tag{26}$$

Here the α_i are weights which ensure that the market equilibrium is efficient. If X_t is aggregate money consumption and $Z^M(\boldsymbol{\pi_t}, X_t)$ is the indirect utility function associated with the market equilibrium, then the efficiency condition is

$$\frac{\partial Z^M(\boldsymbol{\pi}_t, X_t)}{\partial X_t} = \alpha_i \frac{\partial z_i\left(\boldsymbol{\pi}_t, x_{it}\right)}{\partial x_{it}}. \tag{27}$$

The indirect social welfare resulting from one extra pound of consumption is the same whichever household receives it. It follows that households with high marginal utilities of consumption, and thus low consumption are given low weights. It should also be noted that there is no requirement that all households have the same discount rate although if they differ all the wealth in the end will be owned by the household with the lowest discount rate.

Sefton and Weale (2006) (p.246) made use of this result to show that equation (16) holds true for aggregate as well as individual real income. Here aggregate real income is deflated by a plutocratic consumption Divisia index, constructed by weighting individual household price indices by their shares in consumption rather than income. Similarly, the aggregate real rate of interest for the market economy is the sum of the individual real interest rates, weighted by the share of each household in total consumption, $r_t = \sum r_{it} \boldsymbol{\pi}_t' \boldsymbol{c}_{it} / \sum \boldsymbol{\pi}_t' \boldsymbol{c}_{it}$. We set $H_t = \sum_i \frac{p_{it} h_{it}}{P_t^C(0)}$ as the aggregate of exogenous influences on the income of individual households, where $P_t^C(0)$ an aggregate plutocratic price index constructed using consumption weights, and $Q_t^C(0)$ is the corresponding income measure. In appendix B we demonstrate that

$$\begin{aligned} \frac{\dot{Q}_t^C(0)}{Q_t^C(0)} &= \frac{\dot{Y}_t(0)}{Y_t(0)} - \frac{\dot{P}_t^C(0)}{P_t^C(0)} = \frac{r_t}{Y_t \partial Z^M / \partial X_t} \frac{d}{dt} \sum_i \int_t^\infty \alpha_i u_i(\boldsymbol{c}_{i\tau}) e^{-\theta\tau} d\tau \\ &+ \frac{H_t - r_t \int_t^\infty H_\tau e^{-\int_t^\tau r_\nu d\nu} d\tau}{Q_t^C(0)} \end{aligned} \tag{28}$$

[2] Kehoe and Levine (1990) suggest that it is also possible to prove the result when the household discount factors are idiosyncratic. However, in this case the consumption of all households except that with the lowest discount rate would fall to zero in the steady state.

Here the increase in each household's welfare is measured relative to average income and marginal social welfare. But each household's increase in utility is weighted inversely by its marginal utility of consumption. As a result those households with low consumption have low influence on the aggregate. In contrast to equation (25), there is nothing to offset the low weight given to households which have high marginal utilities of consumption.

We complete the picture by noting that $\dot{Q}^C_t(1)/Q^C_t(1) = \dot{Q}_t(1)/Q_t(1)$. In the special case where $\rho = 1$ each household is given equal weight,and it makes no difference whether one is considering a consumption-weighting or an income-weighting framework. But comparison of equations (25) and (28) makes clear the difference between the democratic measure of income growth and the conventional analogue in terms of the way in which they aggregate growth in life-time utility of households.

4.4 Shocks to Income

It is also possible to work out the implications for income and future utility of a shock to income. To the extent that the shock affects future income but not current income, the discounted sum of future utility will increase without any increase in current income, while a temporary shock to current income will result in an increase in discounted utility as well as a temporary increase in income. To understand the effects of shocks to current and future income, we rearrange equation (22),

$$\frac{\frac{d}{dt}\int_t^\infty u_i(\boldsymbol{c}_{i\tau})e^{-\theta\tau}d\tau}{p_{it}\frac{\partial z_{it}}{\partial x_{it}}} = (q_{it} - \boldsymbol{\pi}_t'\boldsymbol{c}_{it}/p_{it}) + \int_t^\infty h_{i\tau}e^{-\int_t^\tau r_{i\nu}d\nu}d\tau. \tag{29}$$

We can see immediately that, for unexpected changes to $h_{i\tau}$ realised in period t, $\Delta h_{i\tau,t}$,the disturbance to the rate of change of utility is given as

$$\frac{\Delta\int_t^\infty u_i(\boldsymbol{c}_{i\tau})e^{-\theta\tau}d\tau}{p_{it}\frac{\partial z_{it}}{\partial x_{it}}} = \int_t^\infty \Delta h_{i\tau,t}\; e^{-\int_t^\tau r_{i\nu}d\nu}d\tau. \tag{30}$$

In the particular case where the shock to income occurs only at time t, we have

$$\frac{\Delta\int_t^\infty u_i(\boldsymbol{c}_{i\tau})e^{-\theta\tau}d\tau}{p_{it}\frac{\partial z_{it}}{\partial x_{it}}} = \Delta h_{it,t}. \tag{31}$$

The shock to income is equal to the associated increase in discounted utility valued at the marginal utility of current consumption. Putting this back into equation (23) delivers the expected result

$$\Delta q_{it}/q_{it} = \Delta h_{it}/q_{it}. \tag{32}$$

Thus, in the case of a coincident shock to income, the contribution of the shock to income growth, $\Delta q_{it}/q_{it}$, is equal to its contribution to discounted utility, valued by the marginal utility of consumption and measured as a proportion of existing income. Shocks to future income have, of course, no direct impact on current income although they do affect life-time welfare.

Whatever the time profile of income shocks, however, the key message from sections 4.2 and 4.3 is that the conventional aggregate downweights the importance given to households whose marginal utility of consumption is low, while the democratic aggregate gives equal weight to each household.

5 Inequality-averse Price Indices and Constant-Scaling Social Cost of Living Indices

The analysis so far has been set out in terms of Atkinson inequality aversion applied to nominal equivalised household income. This approach is heuristic rather than based on a social welfare function; we have highlighted the difference between changes in inequality structured round a nominal inequality averse-aggregate and those which follow from focusing attention on a real inequality-averse aggregate. We now work from a social welfare function defined in terms of the utility of each household. We show, first of all, that with unit elasticity of substitution, the democratic price index is a constant-scaling cost of living index (Crossley and Pendakur 2010) for a population of households with homothetic but heterogenous preferences. Secondly we show that, to a first-order approximation, a social welfare function can be defined over income and that the growth rate of the democratic price index can again be used to deflate the growth rate of the geometric mean of household income to deliver the growth rate of a democratic index of household real income.

5.1 The Democratic Price Index as a Constant Scaling Social Cost of Living Index

Crossley and Pendakur (2010) define a constant scaling cost of living index as the proportionate change to incomes needed to keep social welfare unchanged after a change in prices. As in their work, the analysis is in this section, carried out in a static context in which it is assumed that all income is consumed. We relax this extreme assumption in the next section. We give a functional form to the indirect utility function, using a household specific price index, $p_i(\boldsymbol{\pi}_t)$. The function, $p_i(\boldsymbol{\pi}_t)$ is homogeneous of degree 1 in prices. Then

$$\begin{aligned} z_i(\boldsymbol{\pi}_t, x_{it}, \sigma) &= \frac{1}{1-\sigma}\left(\frac{x_{it}}{p_i\left(\boldsymbol{\pi}_t\right)}\right)^{1-\sigma} \quad \sigma \neq 1 \\ &= \log\left(\frac{x_i}{p_i\left(\boldsymbol{\pi}_t\right)}\right) \quad \sigma = 1 \end{aligned} \tag{33}$$

We use the parameter σ to represent the elasticity of substitution of the indirect utility function. ρ used in equation (1) and subsequently was, in contrast, the degree of inequality aversion.

Social welfare is the sum of individual welfare, with the superscript, U, indicating that the social welfare function, measuring average welfare, is utilitarian rather than, as in section 4 reflecting inequality aversion defined in nominal terms

$$Z^U(\pi_t, x_{1t}....x_{Nt}, \sigma) = \frac{1}{N(1-\sigma)}\sum_i\left(\frac{x_{it}}{p_i\left(\boldsymbol{\pi}_t\right)}\right)^{1-\sigma} \tag{34}$$

In order to compute the change in the CS-SCOLI in response to changes in prices, we need to show how Z^U is affected by a common scaling factor, λ_t. We write

$$Z^U(\pi, x_{1t}....x_{Nt}, \sigma, \lambda_t) = \frac{1}{N(1-\sigma)}\sum_i\left(\frac{\lambda_t x_{it}}{p_i\left(\boldsymbol{\pi}_t\right)}\right)^{1-\sigma} \tag{35}$$

We now wish to find an expression for the rate of change of $\dot{\lambda}$, such that

$$\dot{\lambda}_t\frac{\partial Z^U}{\partial \lambda_t} + \sum_j \frac{\partial Z^U}{\partial \pi_{jt}}\dot{\pi}_{jt} = 0 \tag{36}$$

This tells us how λ_t needs to change to offset the effects on Z^U of changes in prices.

Rearranging,

$$\dot{\lambda}_t = -\frac{\sum_j \frac{\partial Z^U}{\partial \pi_{jt}} \dot{\pi}_{jt}}{\frac{\partial Z^U}{\partial \lambda_t}} \tag{37}$$

Looking first at the numerator

$$\sum_j \frac{\partial Z^U}{\partial \pi_{jt}} \dot{\pi}_{jt} = -\frac{1}{N} \sum_i \left\{ \left(\frac{\lambda_t x_{it}}{p_i(\boldsymbol{\pi}_t)} \right)^{1-\sigma} \sum_j \frac{\partial p_i(\boldsymbol{\pi}_t)}{\partial \pi_{jt}} \frac{\dot{\pi}_{jt}}{p_i(\boldsymbol{\pi}_t)} \right\} \tag{38}$$

Since

$$\frac{\partial p_i(\boldsymbol{\pi}_t)}{\partial \pi_{jt}} = \omega_{ijt} \text{ and } \sum_j \frac{\omega_{ijt} \dot{\pi}_{jt}}{p_i(\boldsymbol{\pi}_t)} = \frac{\dot{p}_i(\boldsymbol{\pi}_t)}{p_i(\boldsymbol{\pi}_t)} \tag{39}$$

it follows that

$$\sum_j \frac{\partial Z^U}{\partial \pi_{jt}} \dot{\pi}_{jt} = -\frac{1}{N} \sum_i \left(\frac{\lambda_t x_{it}}{p_i(\boldsymbol{\pi}_t)} \right)^{1-\sigma} \frac{\dot{p}_i(\boldsymbol{\pi}_t)}{p_i(\boldsymbol{\pi}_t)} \tag{40}$$

where $p_i(\boldsymbol{\pi}_t)$ can be interpreted as the Divisia price index specific to household i.

Also

$$\frac{\partial Z^U}{\partial \lambda_t} = \frac{1}{N \lambda_t} \sum_i \left(\frac{\lambda_t x_{it}}{p_i(\boldsymbol{\pi}_t)} \right)^{1-\sigma} \tag{41}$$

so that

$$\frac{\dot{\lambda}_t}{\lambda_t} = \frac{\sum_i \left(\frac{\lambda_t x_{it}}{p_i(\boldsymbol{\pi}_t)} \right)^{1-\sigma} \frac{\dot{p}_i(\boldsymbol{\pi}_t)}{p_i(\boldsymbol{\pi}_t)}}{\sum_i \left(\frac{\lambda_t x_{it}}{p_i(\boldsymbol{\pi}_t)} \right)^{1-\sigma}} = \frac{\sum_i \left(\frac{x_{it}}{p_i(\boldsymbol{\pi}_t)} \right)^{1-\sigma} \frac{\dot{p}_i(\boldsymbol{\pi}_t)}{p_i(\boldsymbol{\pi}_t)}}{\sum_i \left(\frac{x_{it}}{p_i(\boldsymbol{\pi}_t)} \right)^{1-\sigma}} \tag{42}$$

giving an expression for the rate of growth of the common scaling cost of living index. The growth in the social cost of living index is a weighted average of each household's growth in its price or cost of living index with the weights reflecting each household's real consumption raised to the power of 1-σ. If $\sigma = 1$ then the growth in each household's price index is given equal weight and the growth rate of the constant scaling cost of living index is the growth rate of the democratic cost of living index described by Prais (1958).

This analysis is set out in terms of the instantaneous flow of utility derived from consumption, rather than the accural of life-time utility derived from income. The main focus of this paper has, however, been on the latter. Rather than consider a quantity analogue at this point, we therefore proceed to consider a social welfare function structured round income. We defer the issue of exploring the quantity analogue to this deflator to the next section where we examine what can be done to structure a price/quantity system around income rather than consumption.

5.2 Welfare defined over Income

The welfare which accrues from income to each household in any given period is the sum of the welfare from consumption and that from saving. The former is given by the indirect utility function as set out above while, if resources are allocated efficiently, the latter is equal to the product of saving and the marginal utility of money (see equation (22)). Thus if each household receives a total income of y_{it} of which x_{it} is consumed we can write the utility accruing from income as

$$z_i^y(\boldsymbol{\pi}_t, x_{it}, y_{it}) = z_i(\boldsymbol{\pi}_t, x_{it}) + (y_{it} - x_{it})\frac{\partial z_i(\boldsymbol{\pi}_t, x_{it})}{\partial x_{it}} \tag{43}$$

with the superscript y distinguishing it from the indirect utility function in terms of consumption alone. The corresponding utilitarian social welfare function, Z^Y, is then the sum of each individual household's utility accruing

$$Z^Y(\boldsymbol{\pi}_t, x_{1t}...x_{Nt}, y_{1t}...y_{Nt}) = \frac{1}{N}\sum_i z_i(\boldsymbol{\pi}_t, x_{it}) + (y_{it} - x_{it})\frac{\partial z_i(\boldsymbol{\pi}_t, x_{it})}{\partial x_{it}} \tag{44}$$

The difficulty with this is, of course, that the social welfare is a function of each household's consumption as well as its income. The degree to which income has to be scaled to offset any increase in prices depends on what happens to consumption. That, however, depends on, among other things, expectations of future prices and incomes.

It is, however, possible to make a first order approximation. Equation (43) is a first-order Taylor expansion of

$$z_{it}(\boldsymbol{\pi}_t, x_{it} + [y_{it} - x_{it}]) = z_i(\boldsymbol{\pi}_t, y_{it}). \tag{45}$$

While, with declining marginal utility of consumption, it will understate the utility accruing to households with high savings rates, there can be little doubt that $z_i(\boldsymbol{\pi}, y_{it})$ is better than $z_i(\boldsymbol{\pi}_t, x_{it})$ as an approximation of $z_i^y(\boldsymbol{\pi}_t, x_{it}, y_{it})$. The approximate utilitarian income-based social welfare function is therefore

$$
\begin{aligned}
Z^Y(\boldsymbol{\pi}_t, y_{1t}...y_{Nt}, \sigma) &= \frac{1}{N}\sum_i \left\{ \frac{1}{1-\sigma}\left(\frac{y_{it}}{p_i(\boldsymbol{\pi}_t)}\right)^{1-\sigma} \right\}; \quad \sigma \neq 1 \\
Z^Y(\boldsymbol{\pi}_t, y_{1t}...y_{Nt}, \sigma) &= \frac{1}{N}\sum_i \log\left(\frac{y_{it}}{p_i(\boldsymbol{\pi}_t)}\right); \qquad \sigma = 1.
\end{aligned} \tag{46}
$$

As we showed in the case of the consumption-based welfare function, this leads to a price index whose growth rate is given as

$$
\frac{\dot{P}_t^Y}{P_t^Y} = \frac{\sum_i \left(\frac{y_{it}}{p_i(\boldsymbol{\pi}_t)}\right)^{1-\sigma} \frac{\dot{p}_{it}}{p_{it}}}{\sum_i \left(\frac{y_i}{p_i(\boldsymbol{\pi}_t)}\right)^{1-\sigma}} \tag{47}
$$

In order to develop a quantity measure of income growth we transform $Z^Y(\boldsymbol{\pi}_t, y_{1t}...y_{Nt}, \sigma)$ into a variable which is homogeneous of degree 1 in total real income as

$$
\begin{aligned}
Q_t^Y &= Z_t^{Y\,\frac{1}{1-\sigma}}; \qquad \sigma > 0, \sigma \neq 1 \\
Q_t^Y &= \exp(Z_t^Y); \ \sigma = 1
\end{aligned} \tag{48}
$$

so that, after taking logs

$$
\begin{aligned}
\frac{1}{Q_t^Y}\left(\sum_i \frac{\partial Q_t^Y}{\partial y_{it}}\dot{y}_i + \sum_j \frac{\partial Q_t^Y}{\partial \pi_{jt}}\dot{\pi}_{jt}\right) &= \frac{1}{(1-\sigma)Z^Y}\left(\sum_i \frac{\partial Z^Y}{\partial y_{it}}\dot{y}_{it} + \sum_j \frac{\partial Z^Y}{\partial \pi_{jt}}\dot{\pi}_{jt}\right) \\
&= \frac{\sum_i \left\{ \left(\frac{y_{it}}{p_i(\boldsymbol{\pi}_t)}\right)^{1-\sigma} \frac{\dot{y}_{it}}{y_{it}} - \left(\frac{y_{it}}{p_i(\boldsymbol{\pi}_t)}\right)^{1-\sigma} \sum_j \frac{\partial p_i(\boldsymbol{\pi}_t)}{\partial \pi_{jt}} \frac{\dot{\pi}_{jt}}{p_i(\boldsymbol{\pi}_t)} \right\}}{\sum_i \left(\frac{y_{it}}{p_i(\boldsymbol{\pi}_t)}\right)^{1-\sigma}}
\end{aligned} \tag{49}
$$

Using equation (39)

$$
\begin{aligned}
\frac{1}{Q_t^Y}\left(\sum_i \frac{\partial Q_t^Y}{\partial y_{it}}\dot{y}_{it} + \sum_j \frac{\partial Q_t^Y}{\partial \pi_{jt}}\dot{\pi}_{jt}\right) &= \frac{\sum_i \left\{ \left(\frac{y_{it}}{p_i(\boldsymbol{\pi}_t)}\right)^{1-\sigma} \frac{\dot{y}_{it}}{y_{it}} - \left(\frac{y_{it}}{p_i(\boldsymbol{\pi}_t)}\right)^{1-\sigma} \frac{\dot{p}_i(\boldsymbol{\pi}_t)}{p_i(\boldsymbol{\pi}_t)} \right\}}{\sum_i \left(\frac{y_{it}}{p_i(\boldsymbol{\pi}_t)}\right)^{1-\sigma}} \\
&= \frac{\sum_i q_{it}^{1-\sigma} \frac{\dot{q}_{it}}{q_{it}}}{\sum_i q_{it}^{1-\sigma}}
\end{aligned} \tag{50}
$$

where $q_{it} = y_{it}/p_i(\boldsymbol{\pi})$ is household real income. We can see that this, not surprisingly, leads to a system of value, volume and price indicators very similar to that found when the starting point was Atkinson's inequality aversion. The difference is that the household weights are computed from real household income, while working from Atkinson inequality aversion yielded weights based on nominal income. From a practical point of view, weights based on nominal household income can be compiled from a data source such as a consumer expenditure survey which also provides details of household incomes, as does, for example the United Kingdom's Living Costs and Food Survey. It is much less clear how real income weights could be calculated without either access to a household panel survey or by means of an econometric model such as that used by Crossley and Pendakur (2010) or Jorgenson and Slesnick (2014); the first is, in many cases not practical while the second may, as we noted earlier, not be appealing to government statistical offices.

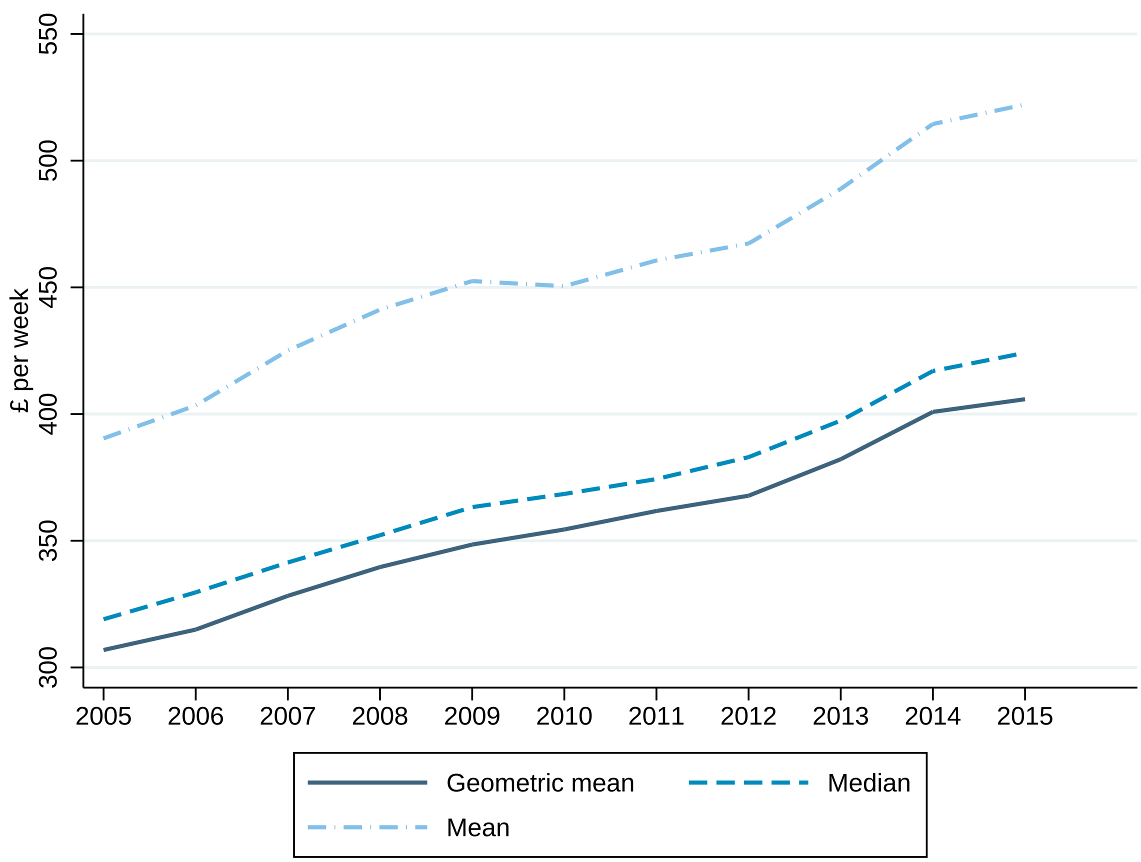
Nevertheless, as before, in the special case where $\sigma = 1$, all households are given equal weights, so the distinction between nominal and real weights does not arise, and the issue of calculating the weights drops away. Layard, Nickell, and Mayraz (2008), on the basis of a number of surveys of happiness, find values of σ between 1.19 and 1.34 with an overall estimate of 1.26, suggesting that the assumption of $\sigma = 1$ is a reasonable approximation given the simplicity it brings both to the calculations and to an explanation of the resulting variables. With the price index defined in this way, the growth in the volume index is equal to the average of each household's growth in money income, less the growth in a price index constructed along the democratic lines suggested by Prais (1958). The price/quantity index system constructed by giving each household's growth experience equal weight can be justified either heuristically as representing a particular form of inequality aversion, or deduced from a welfare function which is a first-order approximation to the utility accruing to each household as a result of its current income.

6 A Democratic Measure of Real Household Income Growth for the United Kingdom

6.1 The Geometric Mean and the Median

Jorgenson and Schreyer (2017) make the point that, if equivalised consumption or income is log-normally distributed, then the median and the geometric mean will co-incide. In fact this is true if the distribution of log consumption or income takes any symmetric distribution. Figure 1 shows the mean, median and geometric mean of equivalised household income after housing costs taken from the Households Below Average Income (HBAI) data set (Department of Work and Pensions 2017).[3] This shows that, in the United Kingdom at least, the geometric mean is indeed close to the median. Between financial years 2005/6 and 2015/6 the nominal geometric mean grew by 32.2 per cent while the median grew by 33.0 per cent and the mean grew by 33.8 per cent.[4] Thus, over this period, at least, there were material differences in the growth rates of the three summary measures, notwithstanding the observation that, in level terms, the geometric mean of income after housing costs was close to the median.

Figure 1: Measures of Central Tendency (£ per week) for UK Equivalised Household Incomes after Housing Costs (Financial Years)



[3]Despite the name, HBAI covers the entire population, and not just those below average income. See Appendix C for more details on this, and other data we use.

[4]The main results we present are computed giving each household its sampling weight in HBAI. However there is a question of how appropriate this is when the geometric mean is used, and we therefore also present unweighted figures for each of the three measures of real income growth in footnote 5.

Figure 2: Growth Rates of the Geometric Mean, Median and Arithmetic Mean of Equivalised Household Income after Housing Costs (Financial Years)

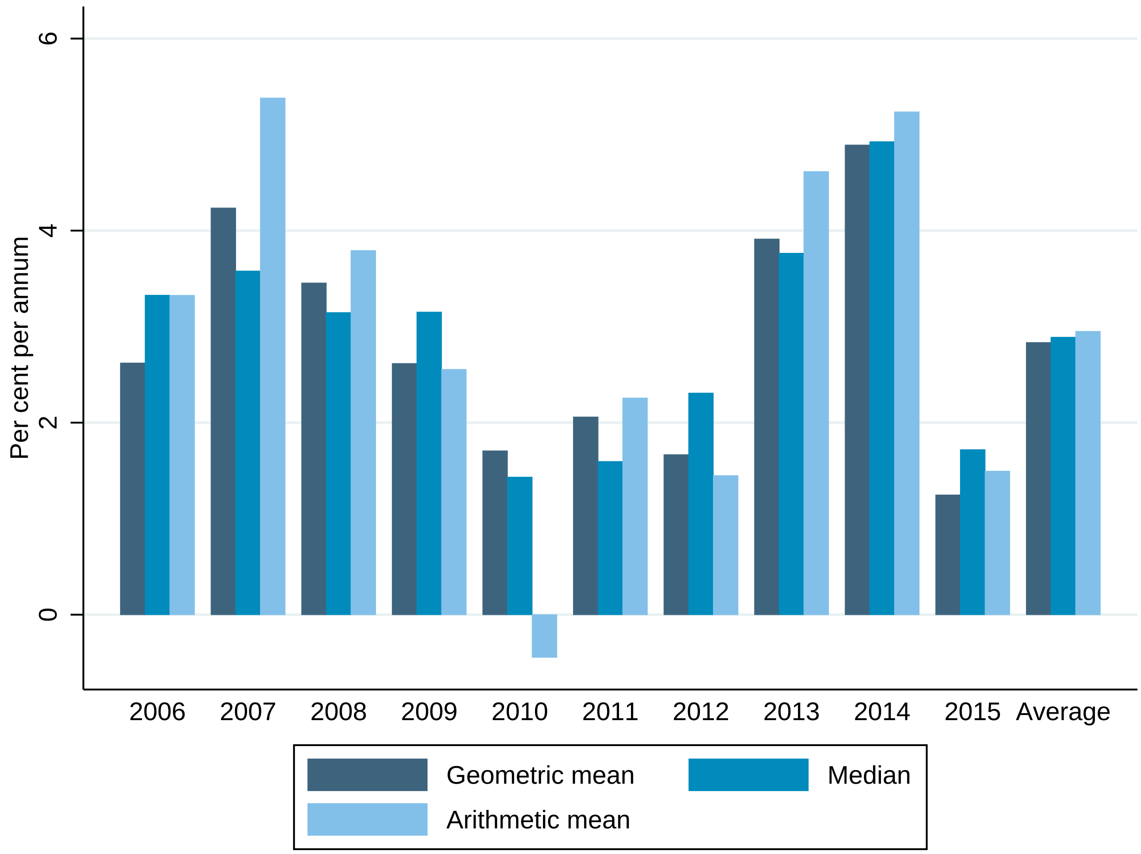


Figure 2 shows the annual growth rates of these three measures of income. There is no general rule as to which rises most or least, but, averaged over the ten years of the data set, the geometric mean has grown at an annual rate of 2.83 per cent, the median at 2.89 per cent and the arithmetic mean at 2.95 per cent per annum. Thus, while in level terms, the median is fairly close to the geometric mean, in growth rate terms, the experience of the median household was midway between the two means.

6.2 Plutocratic and Democratic Price Indices

Since early in 2017 the core measure of inflation in the UK, CPIH has included all housing costs including the imputed rent of owner occupiers; CPI excluding imputed rent is also published. The Office for National Statistics (2017b) presents a measure which excludes imputed and actual rent, maintenance costs and water and sewerage charges as the most suitable measure consistent with conventional CPI for deflating income after housing costs. We use this as our plutocratic price index noting that it is constructed using consumption weights. Our plutocratic price index therefore corresponds to $P^C(0)$ of section 4.3 rather than $P(0)$ of section 3 because this is the price index in common use.

The weights needed for a democratic price index have to be taken from household survey data, while the weights for the plutocratic measure are those of aggregate consumption patterns as shown in the national accounts. Much work on the production of democratic price indices has taken survey data as the sole source from which to calculate the weights for a democratic consumption price index. It is, however, increasingly recognised that there are material differences between expenditure patterns in households surveys and those shown in the national accounts. A standard means of addressing this is to scale survey consumption figures to align them with the national accounts. Thus, for example, Jorgenson and Schreyer (2017) split consumption into fifteen categories and scale each of these before they calculate consumption price indices for households in different parts of the income distribution. Ley (2005) suggests that the difference between the democratic and the plutocratic price indices is likely to depend on how fine the disaggregation is; rich and poor may buy similar amounts in broad categories, but buy rather different goods when fine disaggregations are studied.

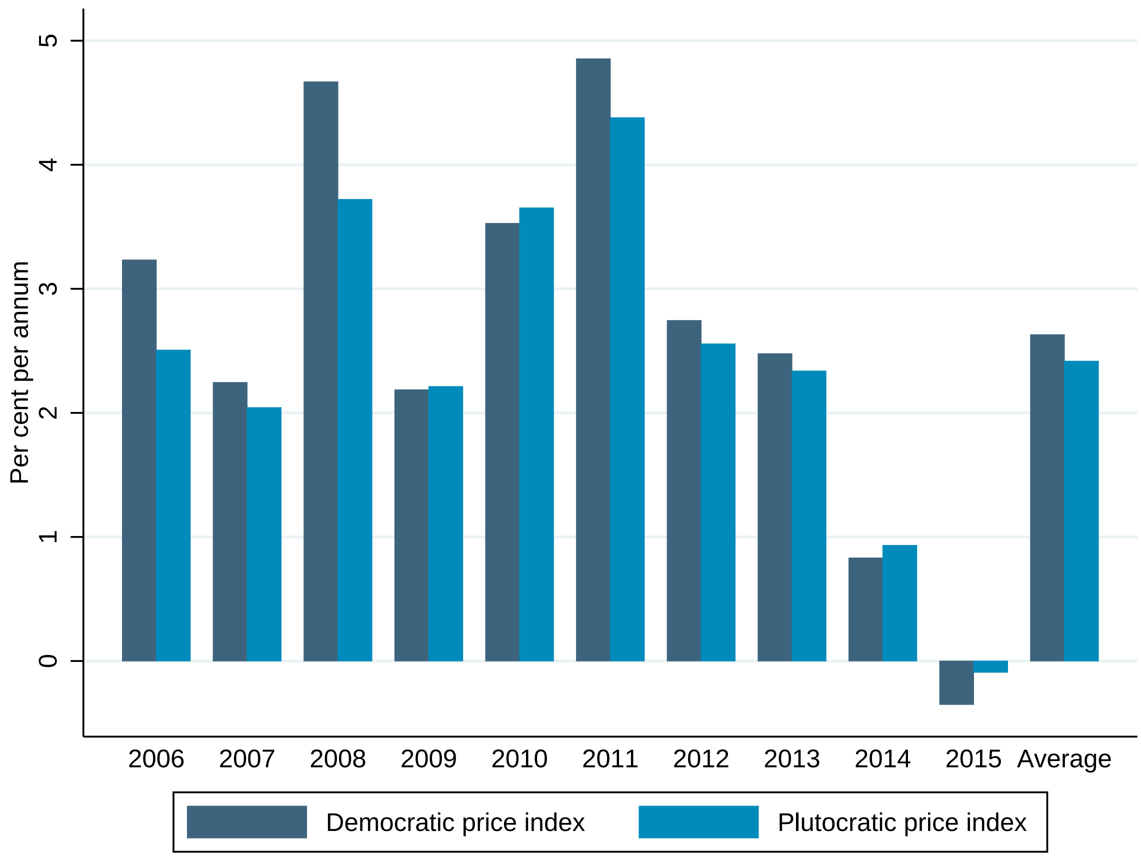
The work on which we draw (Office for National Statistics 2017a) identifies eighty-seven categories of consumption and aligns each of these to the national accounts and thus to the basis used for calculating the conventional plutocratic consumer price index before calculating the mean of the household expenditure shares needed to deliver the democratic price index. The core work in this area led to a price index which included housing costs with owner occupiers' costs measured by imputed rent; however the Office for National Statistics has kindly made available a democratic price index calculated excluding imputed and actual rent, maintenance costs and water and sewerage charges[5] and we have used this as our democratic price index.

Both price indices are chain-linked with the weights updated in every year. There is, however, a concern that erratic movements in consumption may make the use of

[5] Available at https://www.ons.gov.uk/economy/inflationandpriceindices/adhocs/007752democraticmeasureofcpihexcludinghousinguk2005to2016

weights based on the previous year's expenditure patterns alone inappropriate. Instead, therefore, the Office for National Statistics uses the average of the expenditure patterns of the previous three years to provide the weights.

Figure 3: Democratic and Plutocratic Price Inflation excluding Housing Costs (Financial Years)



The annual rates of growth of the two indices are shown in figure 3. Over the ten years from financial year 2005/6 to financial year 2015/6, the democratic price index grew at an average rate of 2.63 per cent per annum while the plutocratic counterpart grew at 2.42 per cent per annum. This difference is well within the range of divergence reported by Ley (2005).

6.3 A Democratic Measure of Equivalised Household Real Income Growth

Figure 4 shows the measures of real income growth calculated from the geometric mean deflated by the democratic price index and the arithmetic mean deflated by the plutocratic price index. The discrepancies in the nominal growth rates are augmented by the differences in the movements of the deflators so, over the ten-year period democratic real household income has grown at 0.20 per cent per annum while plutocratic real household income has grown by 0.52 per cent per annum, and median household income deflated by the plutocratic price index (not shown) has grown at 0.46 per cent per annum.[6] Thus, deflated by the plutocratic price index, growth of median income would give a rather misleading view of the real income growth experience of the average household.

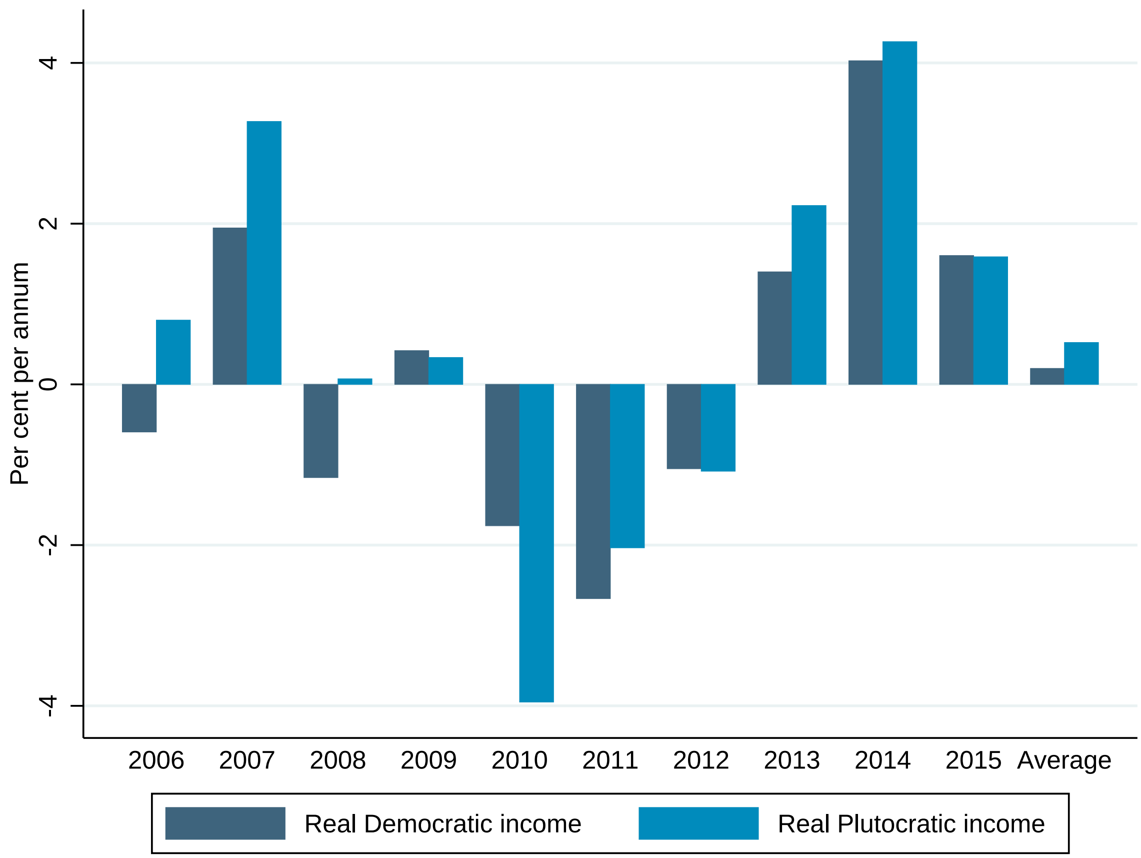
7 Conclusions

This article suggests a practical means by which statistical offices can produce indicators of economic growth which are sensitive to changes in the distribution of resources. Our approach is to focus on the average growth experience of each household. Thus, while the measure points to a focus on the geometric mean rather than the arithmetic mean of real incomes, it can be explained straightforwardly as treating each household's income growth experience equally. We demonstrate that the democratic price index proposed by Prais (1958) is the natural deflator to be applied to the geometric mean. We show that this can be derived from inequality aversion of the type proposed by Atkinson (1970) with the degree of inequality aversion set equal to 1, decomposing the nominal aggregate which results into quantity and price terms. We describe the growth in the quantity indicator as a democratic indicator of household real income growth because each household's experience counts equally. While the same approach can be used with degrees of inequality aversion different from one, household-specific weights enter into the calculations raising practical problems if, for example, income and expenditure data are taken from different sources as in the example we present here.

We show that, when capital accumulation is the sole source of income growth, growth

[6] The real growth rates calculated from the unweighted arithmetic mean, median and geometric mean respectively are: 0.38% p.a., 0.42% p.a. and 0.13% p.a. As before the first two are calculated using the plutocratic price index and the last is deflated using the democratic price index.

Figure 4: Growth Rates of Real Democratic and Plutocratic Equivalised Household Income after Housing Costs (Financial Years)



in the quantity indicator can be given an interpretation in terms of growth in life-time utility measured relative to each household's income valued at the marginal utility of money. This contrasts with the conventional measure of real income growth which measures each household's growth in utility with reference to average income, reducing the importance given to growth in utility of households with low incomes.

Finally we show that the democratic price index can be seen as a constant scaling social cost of living index for a set of household with homothetic but not homogenous preferences and with logarithmic utility functions. Making use of existing results on the interpretation of saving we show that to a first approximation, the constant scaling cost of living index approach provides a framework which, with logarithmic utility, delivers the price/quantity breakdown obtained directly from Atkinson inequality aversion. International empirical evidence suggests that the assumption of logarithmic utility is a reasonable approximation. This suggests that a democratic measure of real income growth could be used as an indicator of welfare change which can be explained straightforwardly to non-technical users.

Calculations for the United Kingdom suggest both that the arithmetic mean of income rose faster than the geometric mean and that the democratic price index rose faster than the plutocratic price index. Taking these two together, a plutocratic measure of household income rose at an annual rate of 0.32% per annum faster than a democratic measure over the ten years from 2005.

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A Demand Shifts and the Divisia Index

This appendix provides an example which shows how the Divisia index can be robust to changes in the mix of demand arising as a result of demand shifts as well as price changes. We assume that utility is given by a constant elasticity of substitution utility function

$$u_i = \left(\sum \left(\frac{c_{ij}}{\psi_{ij}} \right)^{1-\sigma} \right)^{\frac{1}{1-\sigma}} \qquad \sigma > 0;\ \sigma \neq 1 \tag{51}$$

Here ψ_{ij} are the demand parameters which are specific to household i and time-varying. The associated price index is then given as

$$p_i^{\frac{\sigma-1}{\sigma}} = \sum_j (\pi_j \psi_{ij})^{\frac{\sigma-1}{\sigma}} \tag{52}$$

The associated expenditure shares are

$$\omega_{ij} = \frac{(\pi_j \psi_{ij})^{\frac{\sigma-1}{\sigma}}}{\sum_j (\pi_j \psi_{ij})^{\frac{\sigma-1}{\sigma}}} \tag{53}$$

Differentiating (52) yields

$$\frac{dp_i}{p_i} p_i^{1-\sigma} = \sum_j \frac{d\pi_j}{\pi_j} (\pi_j \psi_{ij})^{1-\sigma} + \sum_j \frac{d\psi_{ij}}{\psi_{ij}} (\pi_j \psi_{ij})^{1-\sigma} \tag{54}$$

so that

$$\frac{dp_i}{p_i} = \sum_j \omega_{ij} \frac{d\pi_j}{\pi_j} + \sum_j \omega_{ij} \frac{d\psi_{ij}}{\psi_{ij}} \tag{55}$$

The condition for the Divisia index to be valid for household i is therefore

$$\sum_j \omega_{ij} \frac{d\psi_{ij}}{\psi_{ij}} = 0 \tag{56}$$

The proportionate rates of change in the demand parameters, weighted by the associated expenditure shares, have to sum to zero.

B Welfare and Aggregate Income

We start with the definition of aggregate real income, using the superscript C because the underlying price deflator is calculated using shares of total consumption, weighting each household's consumption shares by its total consumption rather than its total income. In this calculation zero inequality aversion is assumed

$$Q_t^C(0) = \frac{\sum_i p_{it} q_{it}}{P_t^C(0)} \tag{57}$$

where the aggregate price index is defined as

$$\frac{\dot{P}_t^C(0)}{P_t^C(0)} = \sum_i \frac{\boldsymbol{\pi}_t' \boldsymbol{c}_{it}}{\sum_{i,j} \boldsymbol{\pi}_t' \boldsymbol{c}_{it}} \frac{\dot{p}_{it}}{p_{it}}. \tag{58}$$

The rate of change of $Q_t^C(0)$ is

$$\dot{Q}_t^C(0) = \sum_i \left\{ \frac{\dot{p}_{it}}{P_t^C} q_{it} + \frac{\dot{q}_{it}}{P_t^C} p_{it} \right\} - \frac{\dot{P}_t^C}{P_t^C} \frac{\sum_i p_{it} q_{it}}{P_t^C} \tag{59}$$

We now substitute $\dot{q}_{it} = r_{it}(q_{it} - \boldsymbol{\pi}_t' \boldsymbol{c}_{it}/p_{it}) + h_{it}$ to give

$$\dot{Q}_t^C(0) = \sum_i \left\{ \frac{\dot{p}_{it}}{P_t^C(0)} q_{it} + \frac{r_{it}(q_{it} - \boldsymbol{\pi}_t' \boldsymbol{c}_{it}/p_{it}) + h_{it}}{P_t^C(0)} p_{it} \right\} - \frac{\dot{P}_t^C(0)}{P_t^C(0)} \frac{\sum_i p_{it} q_{it}}{P_t^C(0)} \tag{60}$$

First we note that

$$\sum_i \frac{r_{it} \boldsymbol{\pi}_t' \boldsymbol{c}_{it}/p_{it}}{P_t^C(0)} p_{it} = r_t C_t$$

where C_t is an index of aggregate consumption measured as $C_t = \boldsymbol{\pi}_t' \boldsymbol{c}_t / P_t^C(0)$ and r_t is the aggregate real rate of interest defined as the nominal rate of interest less the rate of growth of the aggregate consumption price index. $r_t = r_t^* - \frac{\dot{P}_t^C(0)}{P_t^C(0)}$. For a proof of this see Sefton and Weale (2006), pp 245-246. We can then write (60), putting $H_t = \sum_i \frac{p_{it} h_{it}}{P_t^C(0)}$

as the exogenous contribution to aggregate growth

$$\begin{aligned}
\dot{Q}_t^C(0) &= \sum_i \left\{ \frac{\dot{p}_{it}}{P_t^C(0)} + \frac{r_{it}p_{it}}{P_t^C(0)} - \frac{\dot{P}_t^C(0)}{P_t^C(0)}\frac{p_{it}}{P_t^C(0)} \right\} q_{it} - r_t C_t + H_t \qquad (61) \\
&= \sum_i \left\{ \frac{\dot{p}_{it}}{P_t^C(0)} + \frac{\left(r_t^* - \dot{p}_{it}/p_{it}\right) p_{it}}{P_t^C(0)} - \frac{\dot{P}_t^C(0)}{P_t^C(0)}\frac{p_{it}}{P_t^C(0)} \right\} q_{it} - r_t C_t + H_t \\
&= \sum_i \left\{ \frac{r_t^* p_{it}}{P_t^C(0)} - \frac{\dot{P}_t^C(0)}{P_t^C(0)}\frac{p_{it}}{P_t^C(0)} \right\} q_{it} - r_t C_t + H_t \\
&= \left(r_t^* - \frac{\dot{P}_t^C(0)}{P_t^C(0)} \right) \sum_i \frac{p_{it}q_{it}}{P_t^C(0)} - r_t C_t + H_t \\
&= r_t \left(Q_t^C(0) - C_t \right) + H_t
\end{aligned}$$

This yields the aggregate equivalent of equation (18). We can integrate it to give

$$\begin{aligned}
Q_t^C(0) + \int_t^\infty H_\tau e^{-\int_t^\tau r_{i\nu} d\nu} d\tau &= \int_t^\infty r_\tau \boldsymbol{\pi}_\tau' \boldsymbol{c}_\tau / P_t^C e^{-\int_t^\tau r_\nu d\nu} d\tau \qquad (62) \\
&= \int_t^\infty r_\tau C_t e^{-\int_t^\tau r_\nu d\nu} d\tau
\end{aligned}$$

showing the relationship between aggregate real income and current and future consumption.

We how turn to the growth in utility starting with the equation for household i

$$\frac{d}{dt}\int_t^\infty u_i(\boldsymbol{c}_{i\tau}) e^{-\theta\tau} d\tau = p_{it}\frac{\partial z_{it}}{\partial x_{it}} \int_t^\infty (r_{i\tau} \boldsymbol{\pi}_\tau' \boldsymbol{c}_{i\tau} e^{-\int_t^\tau r_{i\nu} d\nu} / p_{i\tau}) d\tau - \boldsymbol{\pi}_t' \boldsymbol{c}_{it} / p_{it}. \qquad (63)$$

Aggregating using the function (26) and then using equation (27)

$$\begin{aligned}
\frac{d}{dt}\sum_i \int_t^\infty \alpha_i u_i(\boldsymbol{c}_{i\tau}) e^{-\theta\tau} d\tau &= \sum_i p_{it}\alpha_i \frac{\partial z_{it}}{\partial x_{it}} \int_t^\infty (r_{i\tau} \boldsymbol{\pi}_\tau' \boldsymbol{c}_{i\tau} e^{-\int_t^\tau r_{i\nu} d\nu} / p_{i\tau}) d\tau - \boldsymbol{\pi}_t' \boldsymbol{c}_{it} / p_{it} \\
&= \frac{\partial Z^M}{\partial X_t} \sum_i p_{it} \int_t^\infty (r_{i\tau} \boldsymbol{\pi}_\tau' \boldsymbol{c}_{i\tau} e^{-\int_t^\tau r_{i\nu} d\nu} / p_{i\tau}) d\tau - \boldsymbol{\pi}_t' \boldsymbol{c}_{it} / p_{it} \\
&= \frac{\partial Z^M}{\partial X_t} \sum_i p_{it} \int_t^\infty (r_{i\tau} \boldsymbol{\pi}_\tau' \boldsymbol{c}_{i\tau} e^{-\int_t^\tau r_{i\nu} d\nu} / p_{i\tau}) d\tau - \boldsymbol{\pi}_t' \boldsymbol{c}_{it} / p_{it} \\
&= P_t^C(0) \frac{\partial Z^M}{\partial X_t} \left(Q_t^C(0) + \int_t^\infty H_\tau e^{-\int_t^\tau r_\nu d\nu} d\tau - C_t \right) \qquad (64)
\end{aligned}$$

From this it follows that

$$Q_t^C(0) - C_t = \frac{1}{P_t^C \partial Z^M / \partial X_t} \frac{d}{dt} \sum_i \int_t^\infty \alpha_i u_i(\boldsymbol{c}_{i\tau}) e^{-\theta\tau} d\tau - r_t \int_t^\infty H_\tau e^{-\int_t^\tau r_\nu d\nu} d\tau \tag{65}$$

and therefore that

$$\begin{aligned} \frac{\dot{Q}^C(0)}{Q_t^C(0)} &= \frac{r_t}{P_t^C(0) Q_t^C(0) \partial Z^M / \partial X_t} \frac{d}{dt} \sum_i \int_t^\infty \alpha_i u_i(\boldsymbol{c}_{i\tau}) e^{-\theta\tau} d\tau + \frac{H_t - r_t \int_t^\infty H_\tau e^{-\int_t^\tau r_\nu d\nu} d\tau}{Q_t^C(0)} \\ &= \frac{r_t}{Y_t \partial Z^M / \partial X_t} \frac{d}{dt} \sum_i \int_t^\infty \alpha_i u_i(\boldsymbol{c}_{i\tau}) e^{-\theta\tau} d\tau + \frac{H_t - r_t \int_t^\infty H_\tau e^{-\int_t^\tau r_\nu d\nu} d\tau}{Q_t^C(0)} \end{aligned} \tag{66}$$

As with the democratic measure of income growth, there are two components. The first shows the dividend earned on the increase in the social welfare function, measured relative to aggregate money income multiplied by the marginal utility of money. But aggregate social welfare is the weighted sum of individual utility. Each household's weight is inversely proportional to its marginal utility of consumption and thus increase in the welfare of households with high marginal utility and therefore low utility is downweighted as compared to households with low marginal and high absolute utility.

C Data Appendix

The democratic price index we have used is available from Democratic price index and the plutocratic price index from Plutocratic price index.

In both cases we have calculated price indices for financial years (April to March) as the arithmetic averages of the relevant monthly data.

HBAI (Households Below Average Income) data are derived from the Family Resources Survey (FRS), a representative annual sample of about 20,000 private households in the UK. From the HBAI we use the household net income after housing costs variable *esahcohh*. The data are adjusted for household size and composition by equivalisation using the rescaled modifed OECD scale (variable *eqoahchh*). Income is adjusted in the HBAI for undercoverage of top incomes by using replacement values from the Survey of Personal Incomes (SPI). As with any survey, HBAI data are at risk from systematic bias due to non-response by households selected for interview in the FRS. In an attempt to correct for differential non-response, estimates are weighted using population totals.

The analysis is conducted at the household level, and therefore the data are weighted using the household weight *gs_newhh*.